PHYS 1901 Assignment 2

Due: Tuesday, November 21, 2017

DON'T PANIC

This assignment looks tough, but once again, the bark is worse than the bite. You will get a chance to work through some of the details of the Earth's greenhouse effect and the interior of Jupiter. **Please don't delay trying the assignment! Ask for help if you need it!!**

1. **The temperature of Earth without an atmosphere:** The Sun outputs $L_{\mathrm{Sun}} = 3.9 \times 10^{33}$ erg/s. This output is called the solar luminosity and it measures the amount of energy per second that leaves the solar surface as electromagnetic radiation. Here on Earth we receive only a fraction of that power. The flux, or energy per unit area per unit time, diminishes as the square of the distance from the object—the energy gets distributed over a larger and larger area as we move away from the source. At our distance from the sun, the solar flux is: 15 marks

$$f = \frac{L_{\mathrm{Sun}}}{\text{Surface Area of 1AU sphere}} = \frac{L_{\mathrm{Sun}}}{4\pi r_{\mathrm{E-S}}^2}, \tag{1}$$

where $r_{\mathrm{E-S}}$ denotes the Earth-Sun distance. The energy per second that the Earth intercepts from the Sun is the solar flux × the cross-sectional area of the Earth, πR_{Earth}^2. Averaging the intercepted energy per second over the entire Earth's surface leads to the average flux incident on the Earth (notice the extra factor of 4 from averaging),

$$\bar{f} = \frac{L_{\mathrm{Sun}}}{16\pi r_{\mathrm{E-S}}^2}. \tag{2}$$

   (a) Calculate the average flux, $\bar{f}$ incident on the Earth ($r_{\mathrm{E-S}} = 1.5 \times 10^{13}$ cm). Compare your result to the output of a hair dryer, $\bar{f}_{hairdryer} = 1 \times 10^{10}$ erg/s, blowing over 1 square metre.

   (b) The Earth absorbs about 61% of the radiation it receives from the Sun, reflecting the other 39% back into space. The absorbed radiation heats the Earth's surface. Using the Stefan-Boltzmann Law, compute the effective temperature of the Earth's surface: $\sigma T_e^4 = 0.61\bar{f}$; $\sigma = 5.67 \times 10^{-5}$ erg cm^{-2} sec^{-1} K^{-4}. What is this temperature in Celsius? Compare the value of the temperature you computed with the observed value of 290 K. The discrepancy in your comparison is due to the greenhouse effect.

2. **The temperature of Earth with an atmosphere—the greenhouse effect:** In the previous problem, we imagined the Earth without an atmosphere. The absorbed photons heat the Earth's surface which then emits new photons as a blackbody at the 20 marks

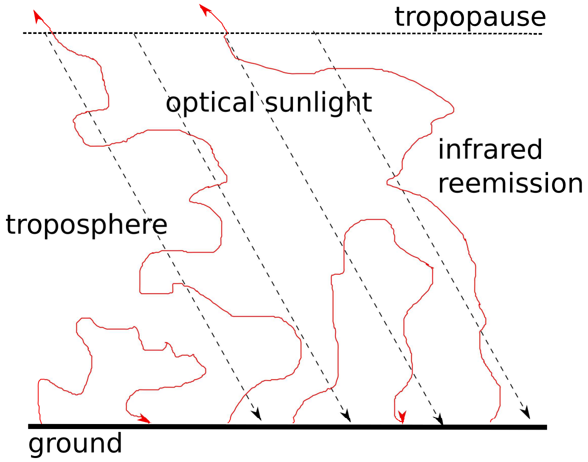


Figure 1: Infrared reemission from the Earth's surface through the troposphere. Infrared scattering (random walking through the troposphere) by water vapour and carbon dioxide leads to the greenhouse effect.

effective temperature, T_e. Most of these photons are in the infrared part of the spectrum. Recall that materials have different optical properties at different frequencies. The atmosphere is no exception—it is transparent to visible light but relatively opaque to infrared radiation. Thus, instead of the surface emitting the infrared blackbody photons directly into space, the infrared photons randomly walk their way through the troposphere before escaping. The randomly walking photons keep the atmosphere of the Earth warmer than the value we calculated in the last question (a good thing for life on Earth!).

A photon that randomly walks in a gas has a mean free path length, l, meaning that on average the photon "flies" about a distance of l before interacting or bumping into a gas particle. Imagine an infrared photon in the troposphere, but only in one dimension—it can move up $(+l)$ or down $(-l)$ at each time step with equal probability. The average position is zero, the photon has no net drift pushing it up or down. However, the *square* of the displacements does have a net drift—even though on average photons are just as likely to have moved up as down, over a long period of time they will have had a chance to wander a long way from the starting point. The mean square distance, D^2, after N steps is $D^2 = Nl^2$. In three dimensions, the result becomes $3D^2 = Nl^2$. Therefore, the number of steps that a photon needs to take on average to get to get to a height, h, is $N = 3h^2/l^2$. The random walk time is just the total distance covered divided by the speed of light, $t_{\text{walk}} = Nl/c = 3h^2/(lc)$.

Let's estimate the size of the greenhouse effect.

(a) Above the troposphere, the Earth's atmosphere becomes transparent to infrared photons, so infrared photons random walk only below the tropopause. (See Figure 1.) Recall that the flux measures the energy per unit area per unit time. So, the flux emitted into space must be:

$$\begin{aligned} F_{\text{rad}} &= \frac{(\text{height}) \times (\text{radiation energy per unit volume})}{\text{random walk time to cover distance } h} \\ &= \frac{h(aT^4)}{3h^2/(lc)} = \frac{4\sigma T^4}{3}\frac{l}{h} = \frac{4\sigma T^4}{3\tau} \end{aligned}$$

where $\tau = h/l$ denotes the optical depth of the troposphere, and where the radiation constant $a = 4\sigma/c$. But, we also know from the last question that $F_{\text{rad}} = \sigma T_e^4$, where T_e is the effective temperature that you calculated in question 1.

(b) Combine the results above to show that $T^4 = (3/4)\tau T_e^4$.

(c) A more refined analysis (the Eddington approximation in radiation transfer) yields the result:

$$T^4 = \frac{3}{4}\left(\tau + \frac{2}{3}\right) T_e^4. \tag{3}$$

At the bottom of the troposphere (i.e., the Earth's surface), $\tau = 2$. Using the improved result, $T^4 = \frac{3}{4}\left(\tau + \frac{2}{3}\right) T_e^4$, calculate the temperature at the Earth's surface. Compare your answer to the your result in question 1.

(d) Suppose that the optical depth, τ, at the bottom of the troposphere increases by 1%. Again using $T^4 = \frac{3}{4}\left(\tau + \frac{2}{3}\right) T_e^4$, how much would this increase in τ raise the surface temperature of the Earth?

(e) Calculate the temperature at the tropopause, where $\tau = 0$. Notice that the temperature at the top of the troposphere is cooler than the effective temperature, T_e. What is the value of τ when $T = T_e$? The height at which $T_e = T$ is the average location of the origin of the photons emerging from the atmosphere into space, not at the tropopause, where $\tau = 0$.

(f) **Bonus question (5 marks):** How close to the Sun would the Earth have to be so its surface temperature without an atmosphere would equal that of the greenhouse temperature at the bottom of the troposphere? Assume that the Earth absorbs only 61% of the incident radiation. (Hint: re-do question 1, but solve for the Earth-Sun distance with the greenhouse temperature.)

What does all the above have to say about anthropogenic climate change? Water vapor is the most important greenhouse gas on Earth, contributing about 60% of the effect. But at the same time, the amount of water vapor in the atmosphere is controlled by the atmosphere's temperature—the temperature of the atmosphere limits the amount of water vapor that the atmosphere can hold. At a given pressure and temperature, once the air becomes saturated, water vapor will start to condense to form liquid water. We see this effect in daily life with clouds and precipitation. All things remaining equal, a warmer atmosphere will hold more water vapor which will increase the greenhouse effect. Humans have released large amounts of carbon dioxide into the atmosphere since the beginning of the industrial revolution. While carbon dioxide is not as important as water vapor in contributing to the greenhouse effect, it is a non-condensable gas—carbon dioxide does not rain out of the sky. The Earth's surface only slowly absorbs carbon dioxide through chemical reactions involving water and rocks in the Earth's crust. Since the atmosphere only slowly loses carbon dioxide, the human produced carbon dioxide adds to the greenhouse effect, warming the atmosphere (increasing the effective optical depth, τ, in the infrared) which increases the capacity of the atmosphere to hold more water vapor, and which further increases the greenhouse effect. Thus we have a positive feedback mechanism. In addition to the positive feedback mechanism, we also have the possibility of a negative one. As the atmosphere holds more water vapor, we expect to see more cloud formation which increases the amount of reflected sunlight. At present, the Earth reflects about 40% of the sunlight it receives from the sun back into space. If a warmer atmosphere leads to more clouds, it is conceivable that the Earth will start to reflect more of the incident sunlight. Such an effect would serve to partially counteract the greenhouse effect. The myriad of feedback mechanisms resulting from the addition of carbon dioxide and other heat captivating gases to our atmosphere, the balance between them, and the resulting equilibrium (or not!) is an active area of climate research. Perhaps we should proceed cautiously in our carbon burning (see assignment 1)!

One thing is clear for the future of our atmosphere, our oceans, and life on Earth. Over the next billion years, the Sun's luminosity will increase by about 10% (take PHYS 1902 to learn why!). The increased solar flux will raise the temperature of the atmosphere to a level sufficient for water vapor to saturate above the tropopause. Currently, the stratosphere is dry; clouds form in the troposphere. Once the stratosphere starts to become wet from the higher saturation point, ultraviolet radiation will separate water into hydrogen and oxygen. The hydrogen will escape into space. The Earth is destined to lose her oceans. Within a billion years after the 10% solar flux increase, the Earth will have become a lifeless, waterless, desert world. Tatooine, here we come!

3. **The Bohr atom and Jupiter:** Louis de Broglie proposed that every particle with momentum $p = mv$ (mass times velocity) has a probability wave with wavelength $\lambda = h/p = h/(mv)$, where h is Planck's constant ($h = 6.63 \times 10^{-27}$ erg·s.) Let us apply this idea to the hydrogen atom.

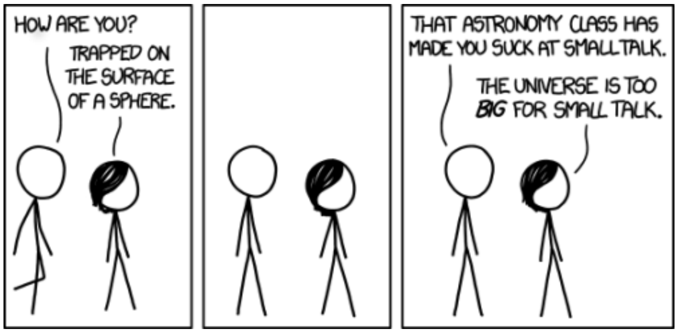
10 marks

   (a) If we treat the electron-proton system classically, the electron moves around the proton in a circle with speed $v_e = \sqrt{e^2/(m_e r)}$, where r is the distance from the proton, e is the electron charge ($e = 4.80 \times 10^{-10}$ electrostatic charge units (esu)), and m_e is the electron mass ($m_e = 9.11 \times 10^{-28}$ gm). If the circumference of an orbit with radius r must be an integer number of de Broglie wavelengths, that is, if $2\pi r =$ (integer)λ_e with $\lambda_e = h/(m_e v_e)$, show that the first allowed orbital distance is $r_B = h^2/((2\pi)^2 m_e e^2)$. Calculate r_B in cm. This the Bohr radius of the hydrogen atom.

   (b) Suppose that we make a cubic lattice of hydrogen atoms where each side of the cube equals a Bohr diameter ($D_B = 2r_B$). Draw a picture to reason that the average density of this cubic lattice of hydrogen is $(m_p + m_e)/(2r_B)^3$, where m_p is the mass of the proton and m_e is the mass of the electron. What is the value of this density?

   Compare the density that you just calculated with the density of Jupiter (look this value up in your textbook)? How might we approximate the interior of Jupiter? Take a look at the BBC Science News article [1].

45 marks



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[1] http://www.bbc.com/news/science-environment-35237985